

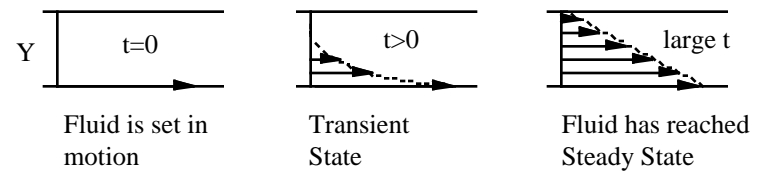
Materials Engineering Science  
MESC. 5025

Instructor: Herve Marand

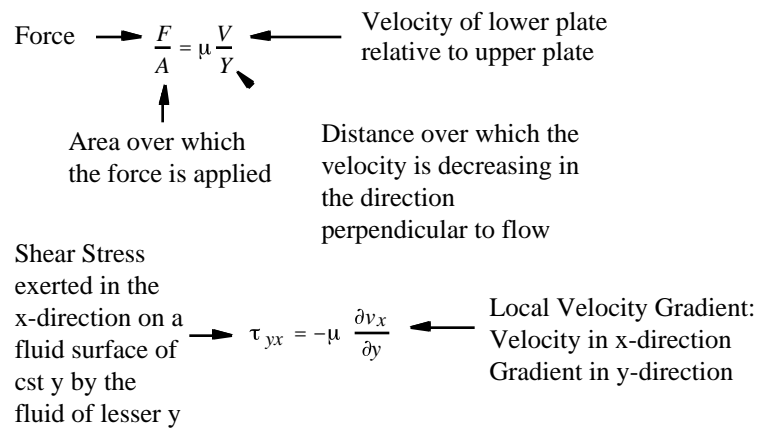
Chapter 5.  
Fluid Flow

### Viscosity and the Mechanism of Momentum Transport

- Viscosity is the physical property which characterizes the resistance of fluids (liquids and gases) to flow.
- Newton's Law of Viscosity  
Consider a fluid contained between two large plates of area  $A$ , separated by a small distance  $Y$ . We examine what happens when the lower plate initially at rest is set in motion at time  $t=0$  with a constant velocity  $V$ . As time proceeds the fluid gains momentum and after some time exhibits a steady-state velocity profile. When this state is reached, a constant force  $F$  is necessary to maintain the velocity of the lower plate at its initial value  $V$ .



### Newton's Law of Viscosity

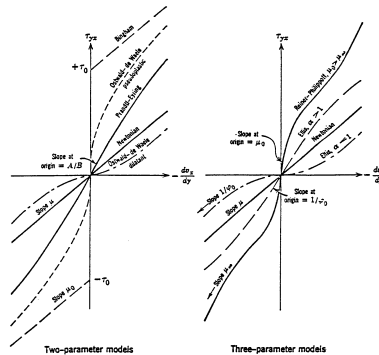


- Note that  $\tau_{yx}$  can be viewed as the viscous flux or the viscous rate of flow of x-momentum in the y-direction
  - The viscous flux is in the direction of negative velocity gradient. Momentum is transferred from a region of high velocity to a region of low velocity. Velocity gradient is the driving force for momentum transfer (analogy with Fourier's law of conduction and Fick's law of diffusion).
  - Kinematic Viscosity  $\nu = \mu/\rho$  where  $\mu$  is the viscosity and  $\rho$  is the density of the fluid
  - Units:
 

	cgs	S.I.
$[\tau_{yx}]$	dyne.cm <sup>-2</sup>	Pa
$[v_x]$	cm.s <sup>-1</sup>	m.s <sup>-1</sup>
$[y]$	cm	m
$[\mu]$	poise	Pa.s
- 1 dyne = 10<sup>-5</sup> N      1 poise = 0.1 Pa.s

## Newtonian versus Non-Newtonian Fluids

- Homogeneous non-polymeric liquids and gases follow the Newtonian behavior (i.e. the coefficient of viscosity,  $\mu$ , is a constant independent of the velocity gradient)
- Fluids that do not follow the Newtonian Behavior are called Non-Newtonian fluids (polymeric liquids, pasts, slurries, particulate dispersions, etc.)



- Generalized Viscosity Law:

$$\tau_{yx} = -\eta \frac{\partial v_x}{\partial y} \quad \text{w/} \quad \eta = f \left( \frac{\partial v_x}{\partial y} \right)$$

if  $\eta$  decreases when  $\frac{\partial v_x}{\partial y}$  increases → Pseudoplastic

if  $\eta$  increases when  $\frac{\partial v_x}{\partial y}$  increases → Dilatant

if  $\eta$  is constant when  $\frac{\partial v_x}{\partial y}$  increases → Newtonian

Other Non-Newtonian behavior under non steady state behavior

Thixotropic: Decrease in  $\eta$  with time after application of stress

Rheopectic: Increase in  $\eta$  with time after application of stress

Viscoelastic: Fluid partially recovers its original shape after removal of stress

### Estimation of Temperature and Pressure Dependence of Fluid Viscosity

- Viscosity of Gases at low density increases with increasing temperature as the momentum of gases is transported by molecules in flight (T increase, kinetic energy and momentum increase, viscosity increases)
- Viscosity of Liquids decreases with increasing temperature as the momentum in liquids is transported through collisions (since molecules only travel short distances) (higher T, larger free volume, less collisions, lower viscosity). Note that in gases of low density, collisions are rare.

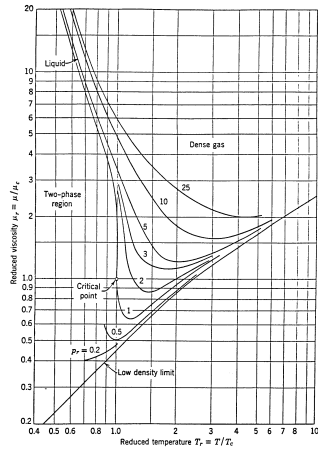
### Principle of Corresponding States

- Estimate  $\mu_C$  either experimentally or empirically (see below)
- For a given T and P, where  $\mu$  needs to be calculated, estimate  $T_R$  and  $P_R$  ( $T_R = T/T_C$  and  $P_R = P/P_C$ )
- Derive  $\mu_R$  from the appropriate curve (next slide)
- Calculate  $\mu$  from  $\mu_R$  and  $\mu_C$  ( $\mu = \mu_R \mu_C$ )
- Watson-Uyehara method for estimation of  $\mu_C$  :

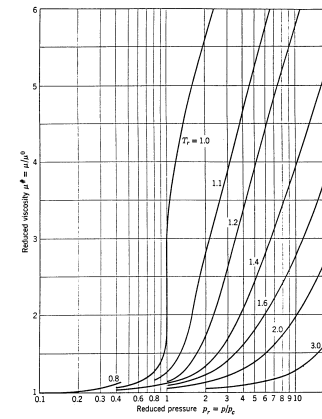
$$\mu_C = 61.6(MT_C)^{1/2} \tilde{V}_C^{-2/3} \quad \text{or} \quad \mu_C = 7.70M^{1/2} P_C^{2/3} T_C^{-1/6}$$

$\mu_C$     in micropoise  
 $P_C$     in atmosphere  
 $T_C$     in Kelvin  
 $V_C$     in  $\text{cm}^3 \cdot \text{g}^{-1}$

Reduced Viscosity as a function of Reduced Temperature for various reduced pressures (from Ueyhara, Watson)



Reduced Viscosity as a function of Reduced Pressure for various reduced temperatures (from Ueyhara, Watson)



### Theory of Gas Viscosity: (low density limit)

1. Kinetic Theory of "Hard Sphere" Gases:

$$\mu = \frac{2}{3\pi^{3/2}} \frac{\sqrt{mkT}}{d^2}$$

← m: molecular mass  
← d: collision diameter

2. Chapman-Enskog Theory:  $\epsilon = 0.77kT_C$

$$V = 4\epsilon \frac{\sigma}{r}^{12} - \frac{\sigma}{r}^6 \quad \sigma = 0.841\tilde{V}_C^{1/3} \text{ or } 2.44 \frac{T_C}{P_C}^{1/3}$$

$$\mu = 2.6693 \cdot 10^{-5} \frac{\sqrt{MT}}{\sigma^2 \mu}$$

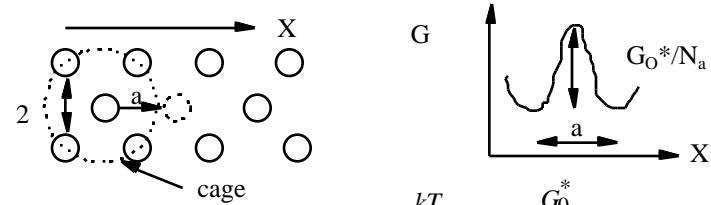
Tabulated for various  $kT/\mu$  values. (Table B2 in BSL)

- T: Kelvin
- P: atmosphere
- M: gram
- V:  $\text{cm}^3 \cdot \text{g}^{-1}$
- : Angstrom
- $\mu$ : poise

### Eyring's Theory of Liquid Viscosity (empirical)

1. Pure Liquid at Rest

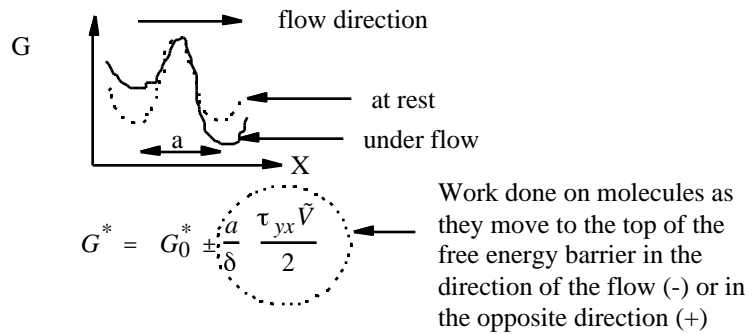
Molecules are in constant motion, which is restricted to displacements within the cage formed by their nearest neighbors. The cage is associated with a free energy barrier of height  $G_0^*/N_a$  where  $N_a$  is Avogadro's number and  $G_0^*$  is the molar free energy barrier. Eyring assumes that the liquid is in dynamic equilibrium: i.e. there are continual rearrangements of the structure by processes where one molecule at a time escapes the cage to fill an adjoining hole.



The jump frequency is:  $\nu = \frac{kT}{h} \exp - \frac{G_0^*}{RT}$

## 2. Pure Liquid under Flow

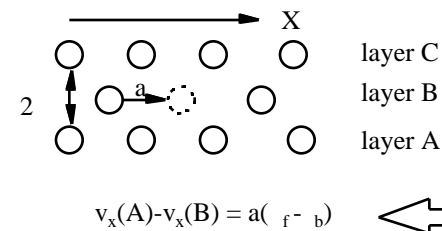
In a fluid of molar volume  $\tilde{V}$  flowing in the X-direction with velocity gradient in the Y direction ( $dv_x/dy$ ), the frequency of molecular rearrangements increases. This is similar to saying that the free energy barrier is distorted when the liquid is under a stress  $\tau_{yx}$ , the new barrier is given by  $G^*/N_a$



If we define  $\nu_f$  to be the frequency of jumps in the forward direction (aided by the stress) and  $\nu_b$  the frequency of jumps in the backward direction (opposing the stress) then:

$$\nu_f = \frac{kT}{h} \exp -\frac{G_0^*}{RT} \exp \frac{a\tau_{yx}\tilde{V}}{2RT\delta} \quad \text{and}$$

$$\nu_b = \frac{kT}{h} \exp -\frac{G_0^*}{RT} \exp \frac{-a\tau_{yx}\tilde{V}}{2RT\delta}$$



The net velocity at which molecules in layer A slip ahead of molecules in layer B is the distance travelled per jump ( $a$ ) multiplied by the net frequency of jumps in the forward direction ( $\nu_f - \nu_b$ )

If we consider that the velocity profile is linear over very small distances between layers, then:

$$\begin{aligned}
 - \frac{\partial v_x}{\partial y} &= - \frac{v_x}{y} = \frac{v_x(A) - v_x(B)}{\delta} \\
 - \frac{\partial v_x}{\partial y} &= \frac{a}{\delta} \frac{kT}{h} \exp \left[ - \frac{G_0^*}{RT} \right] \exp \left[ \frac{\alpha \tau_{yx} \tilde{V}}{2RT\delta} \right] - \exp \left[ - \frac{\alpha \tau_{yx} \tilde{V}}{2RT\delta} \right] \\
 &= 2 \frac{a}{\delta} \frac{kT}{h} \exp \left[ - \frac{G_0^*}{RT} \right] \sinh \left[ \frac{\alpha \tau_{yx} \tilde{V}}{2RT\delta} \right]
 \end{aligned}$$

The velocity gradient is then a non-linear function of the shear stress, thus predicting non-Newtonian behavior. However, for small shear stresses, (small value of X in sinh(X)), one has sinh(X) = X.

$$- \frac{\partial v_x}{\partial y} = \left[ \frac{a}{\delta} \frac{kT}{h} \exp \left[ - \frac{G_0^*}{RT} \right] \right] \tau_{yx} \quad \leftarrow 1/\mu$$

In general  $\alpha$  is approximately unity,  $G_0^*$  is obtained either by fitting the temperature dependence of viscosity or other related physical property or can be estimated crudely through heat of vaporization data at the normal boiling temperature.

$$\begin{aligned}
 G_0^* &= 0.408 U_{\text{vap}} \\
 U_{\text{vap}} &= H_{\text{vap}} - RT_b \\
 H_{\text{vap}} &= T_b S_{\text{vap}} = 10.4 RT_b
 \end{aligned}$$

$$\mu = \frac{N_a h}{\tilde{V}} \exp \left[ 3.8 \frac{T_b}{T} \right]$$

The viscosity of liquids increases exponentially as the temperature is lowered. Calculated in the above fashion you may be about 30% off, which is quite acceptable for a rough estimation.

### Velocity Distribution in Laminar Flow for Simple Geometries and Steady-State Flow

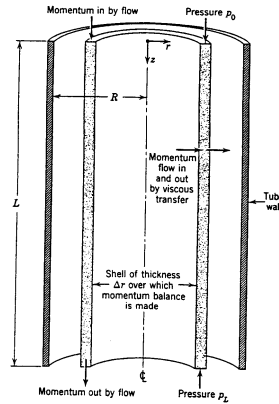
- Using the definition of viscosity and the concept of Force balance for a thin shell of fluid, we can derive the velocity distribution in a steady state rectilinear flow.
- Step #1: Force Balance  
(Rate of momentum in) - (Rate of momentum out) + (Sum of all forces acting on the system) = 0  
This force balance is applied to a thin shell of the fluid of constant thickness
- Step #2: Obtain differential equation  
If we let the thickness approach zero and make use of the definition of a derivative to obtain a differential equation

$$\lim_{\Delta X \rightarrow 0} \frac{Y(X + \Delta X) - Y(X)}{\Delta X} = \frac{dY}{dX}$$

- Step #3: Obtain Velocity Profile  
Inserting the expression for the momentum flux in the differential equation representing the force balance leads to a differential equation for the velocity. Integration of this diff. eq. leads to the velocity profile (and related information: maximum velocity, average velocity, pressure drop, volume rate of flow, forces acting on boundaries, etc...)
- Note that integration of these differential equations will require knowledge of constants of integration (boundary conditions).
  - At solid-fluid interfaces, the fluid velocity equals the velocity at which the solid surface is moving (no-slip condition)
  - At liquid-gas interfaces, the momentum flux (thus the velocity gradient) of the liquid phase can be assumed to be zero
  - At liquid-liquid interfaces, the momentum flux perpendicular to the interface and the velocity are continuous across the interface

### Example #1: Flow Through a Circular Tube

- Use Cylindrical Coordinates, which are the natural coordinates for the description of position in a tube. Consider a steady state laminar flow of a fluid of constant density in a long tube of length  $L$  and radius  $R$  (assume  $L \gg R$  and ignore end effects). Shell of thickness  $\Delta r$  and length  $L$ , considering  $z$  as the flow direction. We now consider the force balance.



### Force Balance

Rate of Momentum In across cylindrical surface at $r$	$[2 r L \tau_z]_r$
Rate of Momentum Out across the cylindrical surface at $r + \Delta r$	$[2 r L \tau_z]_{r + \Delta r}$
Rate of Momentum In across annular surface at $z=0$	$[(2 r r v_z)(v_z)]_{z=0}$
Rate of Momentum Out across annular surface at $z=L$	$[(2 r r v_z)(v_z)]_{z=L}$
Gravity Force	$[2 r r L g]$
Pressure force acting at $z=0$	$[2 r r p_0]$
Pressure force acting at $z=L$	$[2 r r p_L]$

The fluid is incompressible, therefore  $v_z$  is constant in the  $z$ -direction  
 $v_z(z=0) = v_z(z=L)$

Use force balance and divide by  $2Lr$ , then take the limit for  $r \rightarrow 0$ .

This leads to the following differential equation:

$$\frac{d(r\tau_{rz})}{dr} = \frac{P_0 - P_L}{L} r \text{ where } P = p - \rho gz$$

$$r\tau_{rz} = \frac{1}{2} \frac{P_0 - P_L}{L} r^2 + C_1$$

$$\tau_{rz} = \frac{1}{2} \frac{P_0 - P_L}{L} r + \frac{C_1}{r}$$

Since the momentum flux must remain finite,  $C_1$  must be equal to 0

$$\tau_{rz} = \frac{P_0 - P_L}{2L} r$$